Sem V MJC 08 07/11/2025

Ring Theory (continued)

Impt. Theorem

Q :— Prove that a ring R is without zero divisors iff the cancellation laws hold in R.

Proof Given that R is a ring.

Necessary part

Let ring R is without zero divisors.

By definition of zero divisors, it is not possible to find two non-zero elements of R whose product is zero.

i.e. $ab = 0 \Leftrightarrow a = 0$ or $b = 0$

Let $a, b, c \in R$ and $a \neq 0$, and $ab = ac$.

we have to show that cancellation laws hold in R.

$\because$ $ab = ac \Rightarrow ab - ac = 0$

$\Rightarrow a.(b-c) = 0$

$\Rightarrow b - c = 0$ as $a \neq 0$ and R is without zero divisors.

$\Rightarrow b = c$

$\therefore ab = ac \Rightarrow b = c$. So left cancellation law holds in R.

Again let $ba = ca \Rightarrow ba - ca = 0$

$\Rightarrow (b-c)a = 0$

$\Rightarrow b - c = 0$ as $a \neq 0$ and R is without zero divisors.

$\therefore ba = ca \Rightarrow b = c$

So right cancellation law holds in R.

Thus R is without zero divisors $\Rightarrow$ cancellation laws hold in R.

<u>Conversely</u> Let the cancellation laws hold in R.

Then we show that R is without zero divisors.

Let $a, b \in R$ such that $ab = 0$.

Now $ab = 0, a \neq 0 \Rightarrow ab = a.0 \quad [\because a.0 = 0]$

$\Rightarrow b = 0$, by left cancellation law.

$\therefore ab = 0, a \neq 0 \Rightarrow b = 0.$ —— (1)

Again $ab = 0, b \neq 0 \Rightarrow ab = 0.b \quad [\because 0.b = 0 \; \forall b \in R]$

$\Rightarrow a = 0$, by right cancellation law

$\therefore ab = 0, b \neq 0 \Rightarrow a = 0$ —— (2)

From (1) and (2),

$ab = 0 \Rightarrow a = 0$ or $b = 0$

Hence R is without zero divisors.